Riješeni zadaci iz trigonometrije

Zadatak 5: 4) (str. 66) Izracunaj:

$$\cos\frac{17\pi}{9}\cdot\cos\frac{22\pi}{9}+\sin\frac{10\pi}{9}\cdot\sin\frac{13\pi}{9}$$

Rjesenje: Dakle u ovom zadatku bi trebalo prepoznati neki od adicijskih teorema:

$$\sin(s\pm t)=\sin s\cdot\cos t\pm\cos s\cdot\sin t$$
$$\cos(s\pm t)=\cos s\cdot\cos t\mp\sin s\cdot\sin t$$

Promotrim li zadatak uocavam da je prva i osnovna prepreka potpuno razliciti kutevi, ono sto meni treba promatrajuci gornje identitete jesu isti kutevi u parovima. Dakle moj je zadatak uz pomoc formula redukcije svesti dane kuteve na u parovima iste. Prisjetim se da su formule redukcije:

$$\sin\left(\frac{\pi}{2}+t\right)=\cos t \qquad \cos\left(\frac{\pi}{2}+t\right)=-\sin t$$
$$\sin\left(\frac{\pi}{2}-t\right)=\cos t \qquad \cos\left(\frac{\pi}{2}-t\right)=\sin t$$
$$\sin(\pi+t)=-\sin t \qquad \cos(\pi+t)=-\cos t$$
$$\sin(\pi-t)=\sin t \qquad \cos(\pi-t)=-\cos t$$
$$\sin\left(\frac{3\pi}{2}+t\right)=-\cos t \qquad \cos\left(\frac{3\pi}{2}+t\right)=\sin t$$
$$\sin\left(\frac{3\pi}{2}-t\right)=-\cos t \qquad \cos\left(\frac{3\pi}{2}-t\right)=-\sin t$$

Prvo sto cu uciniti jest izracunat cu glavne mjere svih kuteva koji se pojavljuju u pocetnom izrazu. No prije nego to ucinim pogledajmo za koje kuteve uopce taj postupak moram provesti. Prisjetim li se definicije glavne mjere kuta uocavam da glavnu mjeru kuta moram racunati samo za kuteve vece od $2\pi=\frac{18\pi}{9}$. Dakle to znaci da moram odrediti glavnu mjeru samo jednog kuta i to $\frac{22\pi}{9}$. Racunam:

$$t'=t-\left\lfloor\frac{t}{2\pi}\right\rfloor\cdot 2\pi$$

$$t'=\frac{22\pi}{9}-\left\lfloor\frac{\frac{{}^{11}\cancel{22}\cancel{\pi}^{1}}{9}}{\frac{{}_{1}\cancel{2}\cancel{\pi}_{1}}{1}}\right\rfloor\cdot 2\pi$$

$$t'=\frac{22\pi}{9}-\left\lfloor\frac{11}{9}\right\rfloor\cdot 2\pi$$

$$t' = \frac{22\pi}{9} - \underbrace{\lfloor 1.223 \rfloor}_{1} \cdot 2\pi$$

$$t' = \frac{22\pi}{9} - 1 \cdot 2\pi = \frac{22\pi}{9} - 2\pi$$

Svedem na zajednicki nazivnik:

$$t' = \frac{22\pi}{9} - \frac{18\pi}{9} = \frac{4\pi}{9}$$

Dakle moj pocetni izraz sada izgleda na sljedeci nacin:

$$\cos\frac{17\pi}{9} \cdot \cos\frac{4\pi}{9} + \sin\frac{10\pi}{9} \cdot \sin\frac{13\pi}{9}$$

Pokusajmo sada rastaviti gornje kuteve na zbroj ili razliku tako da ona sadrzi specijalne kuteve $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ i 2π. Zapisimo prvo dane specijalne kuteve u obliku razlomaka s nazivnikom 9. Slijedi:

$$\frac{\pi}{2} = \frac{\frac{\pi}{2}}{1} = \frac{\frac{\pi}{2} \cdot 9}{1 \cdot 9} = \frac{\frac{9\pi}{2}}{9}$$

$$\pi = \frac{\pi}{1} = \frac{\pi \cdot 9}{1 \cdot 9} = \frac{9\pi}{9}$$

$$\frac{3\pi}{2} = \frac{\frac{3\pi}{2}}{1} = \frac{\frac{3\pi}{2} \cdot 9}{1 \cdot 9} = \frac{\frac{27\pi}{2}}{9}$$

$$2\pi = \frac{2\pi}{1} = \frac{2\pi \cdot 9}{1 \cdot 9} = \frac{18\pi}{9}$$

Promotrim li malo dane kuteve uocavam da ih mogu zapisati na sljedeci nacin:

$$\frac{17\pi}{9} = \frac{18\pi}{9} - \frac{\pi}{9} = 2\pi - \frac{\pi}{9}$$

$$\frac{10\pi}{9} = \frac{9\pi}{9} + \frac{\pi}{9} = \pi + \frac{\pi}{9}$$

$$\frac{13\pi}{9} = \frac{9\pi}{9} + \frac{4\pi}{9} = \pi + \frac{4\pi}{9}$$

Kut $\frac{4\pi}{9}$ ne moramo mijenjati jer vidimo da se poklapa sa poslijednjim kutom. Nadalje sada preko formula redukcije koje smo prije napisali vrijedi:

$$\cos\left(\frac{17\pi}{9}\right) = \cos\left(2\pi - \frac{\pi}{9}\right) = \cos\left(0 - \frac{\pi}{9}\right) = \cos\left(-\frac{\pi}{9}\right) = \cos\left(\frac{\pi}{9}\right)$$

$$\sin\left(\frac{10\pi}{9}\right) = \sin\left(\pi + \frac{\pi}{9}\right) = -\sin\left(\frac{\pi}{9}\right)$$

$$\sin\left(\frac{13\pi}{9}\right) = \sin\left(\pi + \frac{4\pi}{9}\right) = -\sin\left(\frac{\pi}{9}\right)$$

Nakon svega toga pocetni izraz prelazi u:

$$\cos\frac{\pi}{9}\cdot\cos\frac{4\pi}{9} + \left(-\sin\frac{\pi}{9}\right)\cdot\left(-\sin\frac{4\pi}{9}\right) =$$

$$= \cos\frac{\pi}{9}\cdot\cos\frac{4\pi}{9} + \sin\frac{\pi}{9}\cdot\sin\frac{4\pi}{9}$$

No sada prepoznajemo formulu $\cos(s-t) = \cos s\cdot\cos t + \sin s\cdot\sin t$, dakle vrijedi:

$$\cos\frac{\pi}{9}\cdot\cos\frac{4\pi}{9} + \sin\frac{\pi}{9}\cdot\sin\frac{4\pi}{9} = \cos\left(\frac{\pi}{9} - \frac{4\pi}{9}\right) = \cos\left(-\frac{{}^{1}\cancel{3}\pi}{\cancel{9}_{3}}\right) = \cos\left(-\frac{\pi}{3}\right)$$

Kako je kosinus parna funkcija vrijedi:

$$\cos\frac{\pi}{9}\cdot\cos\frac{4\pi}{9} + \sin\frac{\pi}{9}\cdot\sin\frac{4\pi}{9} = \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Time je zadatak rijesen!

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<u>Zadatak 16</u>: Ako je $\alpha + \beta = \frac{3\pi}{4}$, $\cos\beta = \frac{3\sqrt{7}}{8}$, $\frac{\pi}{2} < \beta < \pi$, koliko je $\sin\alpha$?

<u>Rjesenje</u>: Najprije probajmo odrediti cemu je jednak kosinus od $\alpha + \beta$:

$$\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$$

$$\cos\left(\frac{3\pi}{4}\right) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$$

$$-\cos\left(\frac{\pi}{4}\right) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$$

$$-\frac{\sqrt{2}}{2} = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$$

Nadalje probajmo odrediti cemu je jednak sinus od $\alpha + \beta$:

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

$$\sin\left(\frac{3\pi}{4}\right) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

$$\sin\left(\frac{\pi}{4}\right) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

$$\frac{\sqrt{2}}{2} = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

Dakle na taj smo nacin dobili sljedeci sustav jednadzbi:

$$\begin{cases} -\dfrac{\sqrt{2}}{2} = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \\ \dfrac{\sqrt{2}}{2} = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \end{cases}$$

Nadalje kako nam je dan iznos od $\cos\beta$ te kvadrant u kojem se kut β nalazi mozemo odrediti $\sin\beta$ preko temeljnog indentiteta trigonometrije:

$$\sin^2\beta + \cos^2\beta = 1$$

$$\sin^2\beta + \left(\frac{3\sqrt{7}}{8}\right)^2 = 1$$

$$\sin^2\beta + \frac{3^2\left(\sqrt{7}\right)^2}{8^2} = 1$$

$$\sin^2\beta + \frac{9 \cdot 7}{64} = 1$$

$$\sin^2\beta + \frac{63}{64} = 1$$

$$\sin^2\beta = 1 - \frac{63}{64}$$

$$\sin^2\beta = \frac{1}{64} \;/\sqrt{\;}$$

$$\sin\beta = \pm\frac{1}{8}$$

Iz podatka u zadatku vidljivo je da se kut β nalazi u drugom kvadrantu sto znaci da sinus mora biti pozitivan, dakle vrijedi $sin\beta = \dfrac{1}{8}$. Sada vrijdnosti za sinus i kosinus kuta β uvrstimo u sustav jednadzbi:

$$\begin{cases} -\dfrac{\sqrt{2}}{2} = \cos\alpha \cdot \dfrac{3\sqrt{7}}{8} - \sin\alpha \cdot \dfrac{1}{8} \\ \dfrac{\sqrt{2}}{2} = \sin\alpha \cdot \dfrac{3\sqrt{7}}{8} + \cos\alpha \cdot \dfrac{1}{8} \end{cases}$$

$$\begin{cases} -\dfrac{\sqrt{2}}{2} = \cos\alpha \cdot \dfrac{3\sqrt{7}}{8} - \sin\alpha \cdot \dfrac{1}{8} \\ \dfrac{\sqrt{2}}{2} = \cos\alpha \cdot \dfrac{1}{8} + \sin\alpha \cdot \dfrac{3\sqrt{7}}{8} \;/\cdot\left(-3\sqrt{7}\right) \end{cases}$$

$$\begin{cases} -\dfrac{\sqrt{2}}{2} = \cos\alpha \cdot \dfrac{3\sqrt{7}}{8} - \sin\alpha \cdot \dfrac{1}{8} \\ \dfrac{\sqrt{2}}{2} \cdot \left(-3\sqrt{7}\right) = \cos\alpha \cdot \dfrac{1}{8} \cdot \left(-3\sqrt{7}\right) + \sin\alpha \cdot \dfrac{3\sqrt{7}}{8} \cdot \left(-3\sqrt{7}\right) \end{cases}$$

$$\begin{cases} -\frac{\sqrt{2}}{2} = \cos\alpha \cdot \frac{3\sqrt{7}}{8} - \sin\alpha \cdot \frac{1}{8} \\ -\frac{3\sqrt{2}\sqrt{7}}{2} = -\cos\alpha \cdot \frac{3\sqrt{7}}{8} - \sin\alpha \cdot \frac{\left(3\sqrt{7}\right)^2}{8} \end{cases}$$

$$\begin{cases} -\frac{\sqrt{2}}{2} = \cos\alpha \cdot \frac{3\sqrt{7}}{8} - \sin\alpha \cdot \frac{1}{8} \\ -\frac{3\sqrt{2}\sqrt{7}}{2} = -\cos\alpha \cdot \frac{3\sqrt{7}}{8} - \sin\alpha \cdot \frac{3^2\left(\sqrt{7}\right)^2}{8} \end{cases}$$

$$\begin{cases} -\frac{\sqrt{2}}{2} = \cos\alpha \cdot \frac{3\sqrt{7}}{8} - \sin\alpha \cdot \frac{1}{8} \\ -\frac{3\sqrt{2}\sqrt{7}}{2} = -\cos\alpha \cdot \frac{3\sqrt{7}}{8} - \sin\alpha \cdot \frac{63}{8} \end{cases}$$

Zbrojimo dane jednadzbe te dobijemo:

$$-\frac{\sqrt{2}}{2} + \left(-\frac{3\sqrt{2}\sqrt{7}}{2}\right) = -\sin\alpha \cdot \frac{1}{8} + \left(-\sin\alpha \cdot \frac{63}{8}\right)$$

$$-\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}\sqrt{7}}{2} = -\sin\alpha \cdot \frac{1}{8} - \sin\alpha \cdot \frac{63}{8}$$

$$-\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}\sqrt{7}}{2} = -\sin\alpha \cdot \frac{64}{8}$$

Na lijevoj strani izlucim $-\frac{\sqrt{2}}{2}$, dok na desnoj pokratim sto se pokratiti dade:

$$-\frac{\sqrt{2}}{2}\left(1 + 3\sqrt{7}\right) = -\sin\alpha \cdot \frac{{}^{8}\cancel{64}}{\cancel{8}_1}$$

$$-\frac{\sqrt{2}}{2}\left(1 + 3\sqrt{7}\right) = -\sin\alpha \cdot 8 \ / \cdot \left(-\frac{1}{8}\right)$$

$$-\frac{\sqrt{2}}{2}\left(1 + 3\sqrt{7}\right) \cdot \left(-\frac{1}{8}\right) = -\sin\alpha \cdot 8 \cdot \left(-\frac{1}{8}\right)$$

Pokratim sto se pokratit dade:

$$-\frac{\sqrt{2}}{2}\left(1 + 3\sqrt{7}\right) \cdot \left(-\frac{1}{8}\right) = -\sin\alpha \cdot {}^{1}\cancel{8} \cdot \left(-\frac{1}{\cancel{8}_1}\right)$$

$$\frac{\sqrt{2}}{16}\left(1 + 3\sqrt{7}\right) = \sin\alpha$$

Time je zadatak rijesen!

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