Rijeseni neki zadaci iz poglavlja 3.2

Prije rijesavanja zadataka prisjetimo se bitnih stvari koje ce nas pratiti tijekom njihovog promatranja.

Tvrdnja: (Osnovni trigonometrijski identiteti)

$$\sin^2 x + \cos^2 x = 1$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x}$$

$$\operatorname{ctg} x = \frac{\cos x}{\sin x}$$

Tvrdnja: (Adicijski teoremi)

$$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$$

$$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$$

$$\operatorname{tg}(x \pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \cdot \operatorname{tg} y}$$

$$\operatorname{ctg}(x \pm y) = \frac{\operatorname{ctg} x \cdot \operatorname{ctg} y \mp 1}{\operatorname{ctg} y \pm \operatorname{ctg} x}$$

Tvrdnja: Formule redukcije

$$\sin\left(\frac{\pi}{2} + t\right) = \cos t \qquad \cos\left(\frac{\pi}{2} + t\right) = -\sin t$$

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t \qquad \cos\left(\frac{\pi}{2} - t\right) = \sin t$$

$$\sin(\pi + t) = -\sin t \qquad \cos(\pi + t) = -\cos t$$

$$\sin(\pi - t) = \sin t \qquad \cos(\pi - t) = -\cos t$$

$$\sin\left(\frac{3\pi}{2} + t\right) = -\cos t \qquad \cos\left(\frac{3\pi}{2} + t\right) = \sin t$$

$$\sin\left(\frac{3\pi}{2} - t\right) = -\cos t \qquad \cos\left(\frac{3\pi}{2} - t\right) = -\sin t$$

Tvrdnja: (Trigonometrijske funkcije dvostrukog kuta)

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\operatorname{tg} 2x = \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x}$$

$$\operatorname{ctg} 2x = \frac{\operatorname{ctg}^2 x - 1}{2 \operatorname{ctg} x}$$

Tvrdnja: Sinus i kosinus polovicnog kuta:

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

Tvrdnja: (Transformacija umnoska u zbroj)

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\cos x \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

Tvrdnja: (Transformacija zbroja u umnozak)

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

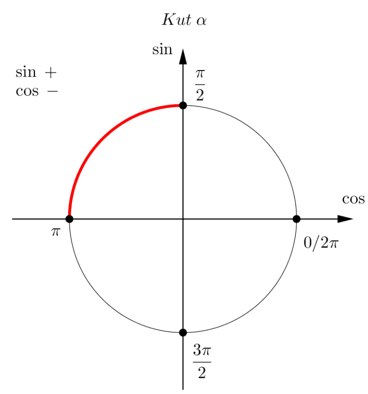
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Prisjetim se jos vrijednosti trigonometrijskih funkcija za istaknute kuteve:

Kut	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos x$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\operatorname{tg} x$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nedef.	0	nedef.	0
$\operatorname{ctg} x$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	nedef.	0	nedef.

Zadatak 7: (str. 72) Koliko je $\operatorname{tg}\frac{\alpha}{2}$ ako je $\operatorname{tg}\alpha = -\frac{24}{7}$, $\frac{\pi}{2} < \alpha < \pi$?

Rjesenje: Uocimo prvo u kojem se kvadrantu nalazi dani kut na brojevnoj kruznici kako bi kasnije znali odrediti predznake vrijednosti trigonometrijskih funkcija:



Nadalje uocimo da je nas zadatak odrediti cemu je jednako $\operatorname{tg}\frac{\alpha}{2}$. Sjetimo li se osnovnih tigonometrijskih identiteta znamo da vrijedi:

$$\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Uvrstimo li $\frac{\alpha}{2}$ umjesto α dobijemo sljedeci izraz:

$$\operatorname{tg}\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

Sada mozemo uociti da je nas zadatak zapravo odrediti cemu je jednako $\sin\frac{\alpha}{2}$ i $\cos\frac{\alpha}{2}$. No znamo da vrijedi sljedece:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Pa dakle mozemo zakljuciti da nam je na kraju zapravo zadatak odrediti cemu je jednak $\cos\alpha$. Nadalje raspisimo ono sto je zadano, $\operatorname{tg}\alpha = -\frac{24}{7}$. Znamo da vrijedi $\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$:

$$\left.\begin{aligned} \operatorname{tg}\alpha &= -\frac{24}{7} \\ \operatorname{tg}\alpha &= \frac{\sin\alpha}{\cos\alpha} \end{aligned}\right\} \quad\Rightarrow\quad \frac{\sin\alpha}{\cos\alpha} = -\frac{24}{7}$$

Pomnozim izraz na desnoj strani s $\cos\alpha$:

$$\frac{\sin\alpha}{\cos\alpha} = -\frac{24}{7} \ / \cdot \cos\alpha$$

$$\sin\alpha = -\frac{24}{7} \cdot \cos\alpha$$

Prisjetimo se da vrijedi temeljni trigonometrijski identitet:

$$\sin^2\alpha + \cos^2\alpha = 1$$

Primjenim cinjenicu da vrijedi $\sin\alpha = -\frac{24}{7} \cdot \cos\alpha$ te dalje racunam:

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\left(-\frac{24}{7} \cdot \cos\alpha\right)^2 + \cos^2\alpha = 1$$

$$\frac{576}{49} \cdot \cos^2\alpha + \cos^2\alpha = 1$$

$$\frac{576}{49} \cdot \cos^2\alpha + \frac{49}{49} \cdot \cos^2\alpha = 1$$

$$\frac{625}{49} \cdot \cos^2\alpha = 1 \ / \cdot \frac{49}{625}$$

$$\cos^2\alpha = \frac{49}{625} \ / \sqrt{\ }$$

$$\cos\alpha = \pm\frac{7}{25}$$

Posto se kut α nalazi u drugom kvadrantu, a tamo su vrijednosti kosinusa negativne, slijedi da je $\cos\alpha = -\frac{7}{25}$. Sada se s tim saznanjem vratim u izraze:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

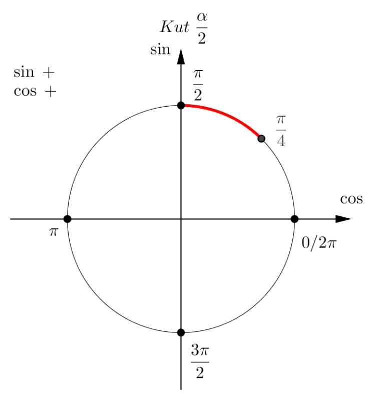
No prije toga moram odrediti gdje se nalazi kut $\frac{\alpha}{2}$ na brojevnoj kruznici kako bi znali odrediti predznake vrijednosti trigonometrijskih funkcija. Kako znam gdje se nalazi kut α, dakle vrijedi $\frac{\pi}{2} < \alpha < \pi$, cijeli izraz podijelim s dva. Racunam:

$$\frac{\pi}{2} < \alpha < \pi \quad / : 2$$

$$\frac{\frac{\pi}{2}}{\frac{2}{1}} < \frac{\alpha}{2} < \frac{\pi}{2}$$

$$\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$$

Na brojevnoj kruznici to izgeda ovako:



Odredimo sada cemu je jednako $\sin\frac{\alpha}{2}$ i $\cos\frac{\alpha}{2}$ imajuci na umu da smo izracunali da vrijedi $\cos\alpha = -\frac{7}{25}$. Dakle racunam prvo cemu je jednako $\sin\frac{\alpha}{2}$:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1+\frac{7}{25}}{2}}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{\frac{25}{25}+\frac{7}{25}}{2}}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{\frac{{}^{16}\cancel{32}}{25}}{\frac{\cancel{2}_1}{1}}} = \pm\sqrt{\frac{\frac{16}{25}}{\frac{1}{1}}} = \pm\sqrt{\frac{16}{25}}$$

$$\sin\frac{\alpha}{2} = \pm\frac{4}{5}$$

Posto se kut $\frac{\alpha}{2}$ nalazi u prvom kvadrantu, a tamo su vrijednosti sinusa pozitivne, slijedi da je $\sin\frac{\alpha}{2}$. Dalje racunam cemu je jednako $\cos\frac{\alpha}{2} = \frac{4}{5}$:

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\left(-\frac{7}{25}\right)}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1-\frac{7}{25}}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{\frac{25}{25}-\frac{7}{25}}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{\frac{{}^{9}\cancel{18}}{25}}{\frac{\cancel{2}_1}{1}}} = \pm\sqrt{\frac{\frac{9}{25}}{\frac{1}{1}}} = \pm\sqrt{\frac{9}{25}}$$

$$\cos\frac{\alpha}{2} = \pm\frac{3}{5}$$

Posto se kut $\frac{\alpha}{2}$ nalazi u prvom kvadrantu, a tamo su vrijednosti kosinusa pozitivne, slijedi da je $\cos\frac{\alpha}{2} = \frac{3}{5}$. Sada kada smo odredili cemu je jednako $\sin\frac{\alpha}{2}$ i $\cos\frac{\alpha}{2}$ mozemo odrediti cemu je jedanko $\operatorname{tg}\alpha 2$. Racunam:

$$\operatorname{tg}\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$\operatorname{tg}\frac{\alpha}{2} = \frac{\frac{4}{{}^{1}\cancel{5}}}{\frac{3}{\cancel{5}_1}} = \frac{\frac{4}{1}}{\frac{3}{1}} = \frac{4}{3}$$

Dakle odredili smo da je $\operatorname{tg} \frac{\alpha}{2} = \frac{4}{3}$ cime je zadatak rijesen.

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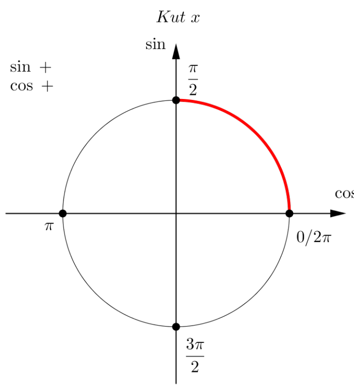
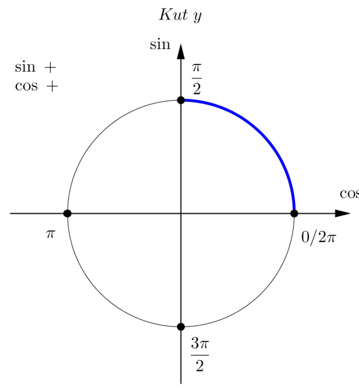
Zadatak 11: (str. 72) Ako je $\sin x = \frac{9}{41}$, $\cos y = \frac{60}{61}$, $x \in \left\langle 0, \frac{\pi}{2} \right\rangle$ $y \in \left\langle 0, \frac{\pi}{2} \right\rangle$, koliko je $\sin^2 \frac{x-y}{2}$?

Rjesenje: Uocimo prvo u kojem se kvadrantu nalaze dani kutevi na brojevnoj kruznici kako bi kasnije znali odrediti predznake vrijednosti trigonometrijskih funkcija:





Nadalje uocimo da je nas zadatak odrediti cemu je jednako $\sin^2 \frac{x-y}{2}$. Sjetimo se iz prethodnog zadatka da vrijedi:

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

Kvadrirajmo taj izraz:

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}$$

Nadalje zamijenimo α u gornjem izrazu s $x - y$, dakle vrijedi:

$$\sin^2 \frac{x-y}{2} = \frac{1 - \cos(x-y)}{2}$$

Mogu uociti da ono sto zapravo moram odrediti jest $\cos(x-y)$. No sjetim se da vrijedi adicijski teorem za kosinus:

$$\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

Dakle kako nam je $\sin x = \frac{9}{41}$, $\cos y = \frac{60}{61}$ zadano u zadatku moramo zapravo odrediti cemu je jednako $\cos x$ u $\sin y$. Odredimo prvo cemu je jednako $\cos x$. Prisjetimo se da vrijedi temeljni trigonometrijski identitet:

$$\sin^2 x + \cos^2 x = 1$$

Primjenim cinjenicu da vrijedi $\sin x = \frac{9}{41}$ te dalje racunam:

$$\sin^2 x + \cos^2 x = 1$$

$$\left(\frac{9}{41}\right)^2 + \cos^2 x = 1$$

$$\frac{81}{1681} + \cos^2 x = 1$$

$$\cos^2 x = 1 - \frac{81}{1681}$$

$$\cos^2 x = \frac{1681}{1681} - \frac{81}{1681}$$

$$\cos^2 x = \frac{1600}{1681} \; /\sqrt{}$$

$$\cos x = \pm\frac{40}{41}$$

Posto se kut x nalazi u prvom kvadrantu, a tamo su vrijednosti kosinusa pozitivne, slijedi da je $\cos x = \frac{40}{41}$.

Nadalje odredimo cemu je jednako $\sin y$ Prisjetimo se da vrijedi temeljni trigonometrijski identitet:

$$\sin^2 y + \cos^2 y = 1$$

Primjenim cinjenicu da vrijedi $\cos y = \frac{60}{61}$ te dalje racunam:

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y + \left(\frac{60}{61}\right)^2 = 1$$

$$\sin^2 y + \frac{3600}{3721} = 1$$

$$\sin^2 y = 1 - \frac{3600}{3721}$$

$$\sin^2 y = \frac{3721}{3721} - \frac{3600}{3721}$$

$$\sin^2 y = \frac{121}{3721} \; /\sqrt{}$$

$$\sin y = \pm\frac{11}{61}$$

Posto se kut y nalazi u prvom kvadrantu, a tamo su vrijednosti sinusa pozitivne, slijedi da je $\sin y = \frac{11}{61}$.
Time smo odredili sve vrijednosti da bi mogli odrediti cemu je jednako $\cos(x-y)$, dakle prisjetim se da vrijedi:

$$\sin x = \frac{9}{41}, \quad \cos x = \frac{40}{41}, \quad \sin y = \frac{11}{61}, \quad \cos y = \frac{60}{61}$$

Te vrijednosti uvrstim u raspisani izraz za $\cos(x-y)$. Racunam:

$$\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

$$\cos(x-y) = \frac{40}{41} \cdot \frac{60}{61} + \frac{9}{41} \cdot \frac{11}{61}$$

$$\cos(x-y) = \frac{2400}{2501} + \frac{99}{2501}$$

$$\cos(x-y) = \frac{2499}{2501}$$

Preostaje nam samo jos odrediti cemu je jedanko $\sin^2\frac{x-y}{2}$. no kako znam da vrijedi:

$$\sin^2\frac{x-y}{2} = \frac{1-\cos(x-y)}{2}$$

Uvrstimo $\cos(x-y) = \frac{2499}{2501}$ u taj izraz. Racunam:

$$\sin^2\frac{x-y}{2} = \frac{1-\cos(x-y)}{2}$$

$$\sin^2\frac{x-y}{2} = \frac{1-\frac{2499}{2501}}{2} = \frac{\frac{2501}{2501}-\frac{2499}{2501}}{2} = \frac{\frac{{}^{1}\cancel{2}}{2501}}{\frac{\cancel{2}_{1}}{1}} = \frac{\frac{1}{2501}}{\frac{1}{1}}$$

$$\sin^2\frac{x-y}{2} = \frac{1}{2501}$$

Dakle odredili smo da je $\sin^2\frac{x-y}{2}$ jednako $\frac{1}{2501}$, te je time zadatak rijesen.

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<u>Zadatak 13</u>: (str. 72) 7) Dokazi identitet:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \operatorname{tg}^2 x$$

Rjesenje: Zadatak nam je pokazati da je lijeva strana izraza jednaka desnoj pa u tu svrhu pokusajmo raspisati lijevu stranu izraza. Prisjetimo se da vrijedi sljedeci identiet:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

Dakle raspisimo prvo $\sin 4x$. Vrijedi:

$$\sin 4x = 2 \sin 2x \cos 2x$$

Uvrstimo tu cinjenicu u lijevu stranu pocetnog izraza, $\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x}$. Dakle racunam:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{2\sin 2x - 2\sin 2x \cdot \cos 2x}{2\sin 2x + 2\sin 2x \cdot \cos 2x}$$

Izlucim $2 \sin 2x$ i u brojniku i u nazivniku danog izraza:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{2\sin 2x - 2\sin 2x \cdot \cos 2x}{2\sin 2x + 2\sin 2x \cdot \cos 2x} = \frac{2\sin 2x\,(1-\cos 2x)}{2\sin 2x\,(1+\cos 2x)}$$

Pokratim sto se pokratiti dade:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{{}^1\cancel{2\sin 2x}\,(1-\cos 2x)}{\cancel{2\sin 2x}_1\,(1+\cos 2x)} = \frac{1-\cos 2x}{1+\cos 2x}$$

Nadalje prisjetim se da vrijede sljedeca dva identiteta, temeljni trigonometrijski identite, te kosinus dvostrukog kuta:

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

Uvrstim te cinjenice u gornji izraz. Slijedi:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{1-\cos 2x}{1+\cos 2x} = \frac{\sin^2 x + \cos^2 x - \left(\cos^2 x - \sin^2 x\right)}{\sin^2 x + \cos^2 x + \cos^2 x - \sin^2 x}$$

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{\sin^2 x + \cos^2 x - \cos^2 x + \sin^2 x}{\sin^2 x + \cos^2 x + \cos^2 x - \sin^2 x}$$

Pokratim sto se pokratiti dade:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{\sin^2 x + \cancel{\cos^2 x} \cancel{- \cos^2 x} + \sin^2 x}{\cancel{\sin^2 x} + \cos^2 x + \cos^2 x \cancel{- \sin^2 x}} = \frac{\sin^2 x + \sin^2 x}{\cos^2 x + \cos^2 x}$$

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{\sin^2 x + \sin^2 x}{\cos^2 x + \cos^2 x} = \frac{2\sin^2 x}{2\cos^2 x}$$

Pokratim sto se pokratiti dade:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{{}^1\cancel{2}\sin^2 x}{\cancel{2}_1\cos^2 x} = \frac{\sin^2 x}{\cos^2 x}$$

Imajuci na umu da vrijedi idenditet $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$ dalje slijedi:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \frac{\sin^2 x}{\cos^2 x} = \left(\frac{\sin x}{\cos x}\right)^2$$

No prema jednom od osnovnih trigonometrijskih identiteta znam da vrijedi $\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$, pa konacno uocavam da vrijedi:

$$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = \left(\frac{\sin x}{\cos x}\right)^2 = (\operatorname{tg} x)^2 = \operatorname{tg}^2 x$$

Pogledamo li desnu stranu izraza iz zadatka vidimo da je ona identicna onome sto smo dobili pa je time zadatak rijesen.

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Zadatak 13: (str. 72) 10) Dokazi identitet:

$$\frac{1 - \sin 2x}{1 + \sin 2x} = \operatorname{ctg}^2\left(\frac{\pi}{4} + x\right)$$

Rjesenje: Zadatak nam je pokazati da je lijeva strana izraza jednaka desnoj pa u tu svrhu pokusajmo raspisati desnu stranu izraza:

$$\operatorname{ctg}^2\left(\frac{\pi}{4} + x\right) = \left(\operatorname{ctg}\left(\frac{\pi}{4} + x\right)\right)^2$$

Prisjetimo se da vrijedi sljedeci adicijski teorem za kotangens:

$$\operatorname{ctg}(\alpha + \beta) = \frac{\operatorname{ctg}\alpha \operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\beta}$$

Probajmo imajuci na umu taj identite odrediti cemu je jednako $\operatorname{ctg}\left(\frac{\pi}{4} + x\right)$. Racunam:

$$\operatorname{ctg}\left(\frac{\pi}{4} + x\right) = \frac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg} x - 1}{\operatorname{ctg}\frac{\pi}{4} + \operatorname{ctg} x}$$

Iscitam iz tablice na prvoj stranici dokumenta da vrijedi $\operatorname{ctg}\frac{\pi}{4} = 1$. Dakle dalje slijedi:

$$\operatorname{ctg}\left(\frac{\pi}{4} + x\right) = \frac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg} x - 1}{\operatorname{ctg}\frac{\pi}{4} + \operatorname{ctg} x} = \frac{1 \cdot \operatorname{ctg} x - 1}{1 + \operatorname{ctg} x} = \frac{\operatorname{ctg} x - 1}{1 + \operatorname{ctg} x}$$

Nadalje prema jednom od osnovnih trigonometrijskih identiteta znam da vrijedi $\operatorname{ctg}\alpha = \dfrac{\cos\alpha}{\sin\alpha}$. Imajuci to na umu dalje racunam:

$$\operatorname{ctg}\left(\frac{\pi}{4}+x\right) = \frac{\operatorname{ctg}x - 1}{1+\operatorname{ctg}x} = \frac{\dfrac{\cos x}{\sin x} - 1}{1 + \dfrac{\cos x}{\sin x}} = \frac{\dfrac{\cos x}{\sin x} - \dfrac{1}{1}}{\dfrac{1}{1} + \dfrac{\cos x}{\sin x}}$$

Izraze u brojniku i nazivniku svedem na zajednicki nazivnik $\sin x$. Slijedi:

$$\operatorname{ctg}\left(\frac{\pi}{4}+x\right) = \frac{\dfrac{\cos x}{\sin x} - \dfrac{1}{1}}{\dfrac{1}{1} + \dfrac{\cos x}{\sin x}} = \frac{\dfrac{\cos x - \sin x}{\sin x}}{\dfrac{\sin x + \cos x}{\sin x}}$$

Pokratim sto se pokratiti dade:

$$\operatorname{ctg}\left(\frac{\pi}{4}+x\right) = \frac{\dfrac{\cos x - \sin x}{{}^{1}\cancel{\sin x}}}{\dfrac{\sin x + \cos x}{\cancel{\sin x}_{1}}} = \frac{\dfrac{\cos x - \sin x}{1}}{\dfrac{\sin x + \cos x}{1}} = \frac{\cos x - \sin x}{\sin x + \cos x}$$

Vratimo se sada na pocetak, drugim rijecima, na izraz $\operatorname{ctg}^2\left(\frac{\pi}{4}+x\right) = \left(\operatorname{ctg}\left(\frac{\pi}{4}+x\right)\right)^2$. Uvrstimo gore odredeni izraz:

$$\operatorname{ctg}^2\left(\frac{\pi}{4}+x\right) = \left(\operatorname{ctg}\left(\frac{\pi}{4}+x\right)\right)^2 = \left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)^2$$

Imajuci na umu da vrijedi idenditet $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$ dalje slijedi:

$$\operatorname{ctg}^2\left(\frac{\pi}{4}+x\right) = \left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)^2 = \frac{(\cos x - \sin x)^2}{(\sin x + \cos x)^2}$$

Raspisem brojnik i nazivnik prema identitetu $(a+b)^2 = a^2 + 2ab + b^2$. Dalje vrijedi:

$$\operatorname{ctg}^2\left(\frac{\pi}{4}+x\right) = \frac{(\cos x - \sin x)^2}{(\sin x + \cos x)^2} = \frac{\cos^2 x - 2\cos x\sin x + \sin^2 x}{\sin^2 x + 2\sin x\cos x + \cos^2 x}$$

Zapisimo to na malo drugaciji nacin, promijenivsi poredak clanovima sume i umnoska. Slijedi:

$$\operatorname{ctg}^2\left(\frac{\pi}{4}+x\right) = \frac{\sin^2 x + \cos^2 x - 2\sin x\cos x}{\sin^2 x + \cos^2 x + 2\sin x\cos x}$$

Nadalje prisjetim se da vrijede sljedeca dva identiteta, temeljni trigonometrijski identite, te sinus dvostrukog kuta:

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Uvrstim te cinjenice u gornji izraz. Slijedi:

$$\operatorname{ctg}^2\left(\frac{\pi}{4}+x\right) = \frac{\overbrace{\sin^2 x+\cos^2 x}^{1} - \overbrace{2\sin x\cos x}^{\sin 2x}}{\underbrace{\sin^2 x+\cos^2 x}_{1} + \underbrace{2\sin x\cos x}_{\sin 2x}} = \frac{1-\sin 2x}{1+\sin 2x}$$

$$\operatorname{ctg}^2\left(\frac{\pi}{4}+x\right) = \frac{1-\sin 2x}{1+\sin 2x}$$

Pogledamo li lijevu stranu izraza iz zadatka vidimo da je ona identicna onome sto smo dobili pa je time zadatak rijesen.

$- \star -$

<u>Zadatak 13</u>: (str. 72) 14) Dokazi identitet:

$$\frac{\sin^4 x + 2\sin x\cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \cos 2x$$

<u>Rjesenje</u>: Zadatak nam je pokazati da je lijeva strana izraza jednaka desnoj pa u tu svrhu pokusajmo raspisati lijevu stranu izraza:

$$\frac{\sin^4 x + 2\sin x\cos x - \cos^4 x}{\operatorname{tg} 2x - 1}$$

Zapisimo clanove sume u brojniku gornjeg razlomka drugim redoslijedom i imajuci na umu da vrijedi $\sin^4 x = \left(\sin^2 x\right)^2$ i $\cos^4 x = \left(\cos^2 x\right)^2$:

$$\frac{\sin^4 x + 2\sin x\cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \frac{\left(\sin^2 x\right)^2 - \left(\cos^2 x\right)^2 + 2\sin x\cos x}{\operatorname{tg} 2x - 1}$$

Sada mogu prepoznati da prva dva clana sume u brojniku mogu raspisati prema identitetu za razliku kvadrata $a^2 - b^2 = (a-b)(a+b)$. Slijedi:

$$\frac{\sin^4 x + 2\sin x\cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \frac{\left(\sin^2 x\right)^2 - \left(\cos^2 x\right)^2 + 2\sin x\cos x}{\operatorname{tg} 2x - 1} =$$

$$= \frac{\left(\sin^2 x - \cos^2 x\right)\left(\sin^2 x + \cos^2 x\right) + 2\sin x\cos x}{\operatorname{tg} 2x - 1}$$

Nadalje prisjetim se da vrijedi osnovni tirgonometrijski identitet:

$$\sin^2\alpha + \cos^2\alpha = 1$$

Dakle pocetni izraz prelazi u:

$$\frac{\sin^4 x + 2\sin x\cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \frac{\left(\sin^2 x - \cos^2 x\right)\overbrace{\left(\sin^2 x + \cos^2 x\right)}^{1} + 2\sin x\cos x}{\operatorname{tg} 2x - 1} =$$

$$= \frac{\left(\sin^2 x - \cos^2 x\right) \cdot 1 + 2\sin x \cos x}{\operatorname{tg} 2x - 1} = \frac{\sin^2 x - \cos^2 x + 2\sin x \cos x}{\operatorname{tg} 2x - 1}$$

Prisjetim se da vrijede sljedeca dva identiteta, za sinus dvostrukog kuta, te za kosinus dvostrukog kuta:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

U tu svrhu izlucim prvo $-$ iz prva dva clana sume u brojniku:

$$= \frac{\left(\sin^2 x - \cos^2 x\right) \cdot 1 + 2\sin x \cos x}{\operatorname{tg} 2x - 1} = \frac{-\left(-\sin^2 x + \cos^2 x\right) + 2\sin x \cos x}{\operatorname{tg} 2x - 1} =$$

$$= \frac{-\left(\cos^2 x - \sin^2 x\right) + 2\sin x \cos x}{\operatorname{tg} 2x - 1} =$$

Primjenim izraze za sinus i kosinus dvostrukog kuta na gornji izraz. Slijedi:

$$\frac{\sin^4 x + 2\sin x \cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \frac{-\overbrace{\left(\cos^2 x - \sin^2 x\right)}^{\cos 2x} + \overbrace{2\sin x \cos x}^{\sin 2x}}{\operatorname{tg} 2x - 1} = \frac{-\cos 2x + \sin 2x}{\operatorname{tg} 2x - 1} =$$

$$= \frac{\sin 2x - \cos 2x}{\operatorname{tg} 2x - 1}$$

Nadalje prisjetim se da vrijedi osnovni trigonometrijski identitet za tangense:

$$\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Imajuci to na umu raspisem nazivnik gornjeg izraza:

$$\frac{\sin^4 x + 2\sin x \cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \frac{\sin 2x - \cos 2x}{\operatorname{tg} 2x - 1} = \frac{\sin 2x - \cos 2x}{\frac{\sin 2x}{\cos 2x} - 1}$$

Svedem izraze u nazivniku na zajednicki nazivnik $\cos 2x$. Racunam:

$$\frac{\sin^4 x + 2\sin x \cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \frac{\sin 2x - \cos 2x}{\frac{\sin 2x - \cos 2x}{\cos 2x}} = \frac{\frac{\sin 2x - \cos 2x}{1}}{\frac{\sin 2x - \cos 2x}{\cos 2x}}$$

Pokratim ono sto se pokratiti dade:

$$\frac{\sin^4 x + 2\sin x \cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \frac{\frac{{}^1\,\cancel{\sin 2x - \cos 2x}}{1}}{\frac{\cancel{\sin 2x - \cos 2x}_{\,1}}{\cos 2x}} = \frac{\frac{1}{1}}{\frac{1}{\cos 2x}} = \frac{\cos 2x}{1}$$

$$\frac{\sin^4 x + 2\sin x\cos x - \cos^4 x}{\operatorname{tg} 2x - 1} = \cos 2x$$

Pogledamo li desnu stranu izraza iz zadatka vidimo da je ona identicna onome sto smo dobili pa je time zadatak rijesen.

$- \star -$

Zadatak 14: (str. 73) 6) Dokazi identitet:

$$\frac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha} = \operatorname{tg}\frac{\alpha}{2}$$

Rjesenje: Zadatak nam je pokazati da je lijeva strana izraza jednaka desnoj pa u tu svrhu pokusajmo raspisati lijevu stranu izraza:

$$\frac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha}$$

Kako se na desnoj strani nalazi izraz koji sadrzi polovican kut α, drugim rijecima $\left(\frac{\alpha}{2}\right)$, zamislimo da umjesto α u nasem izrazu kojeg raspisujem pise $2\frac{\alpha}{2}$. Zapisimo taj izraz tada imajuci na umu tu cinjenicu:

$$\frac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha} = \frac{1 + \sin\left(2\frac{\alpha}{2}\right) - \cos\left(2\frac{\alpha}{2}\right)}{1 + \sin\left(2\frac{\alpha}{2}\right) + \cos\left(2\frac{\alpha}{2}\right)}$$

Prisjetim se da vrijede sljedeca tri identiteta, za sinus dvostrukog kuta, za kosinus dvostrukog kuta, te temeljni trigonometrijski identitet:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

Uvrstim li u njih umjesto α $\frac{\alpha}{2}$ uocavam da vrijede sljedeci izrazi:

$$\sin\left(2\frac{\alpha}{2}\right) = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \quad \Rightarrow \quad \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\cos\left(2\frac{\alpha}{2}\right) = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} \quad \Rightarrow \quad \cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$$

$$\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1$$

Primjenim te identitete na gornji izraz. Slijedi:

$$\frac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha} = \frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right)}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}} =$$

$$= \frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}$$

Pokratim sto se pokratiti dade:

$$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha+\cos\alpha} = \frac{\sin^2\frac{\alpha}{2} + \cancel{\cos^2\frac{\alpha}{2}} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \cancel{- \cos^2\frac{\alpha}{2}} + \sin^2\frac{\alpha}{2}}{\cancel{\sin^2\frac{\alpha}{2}} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} \cancel{- \sin^2\frac{\alpha}{2}}} =$$

$$= \frac{\sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}} = \frac{2\sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

Izlucim $2\sin\frac{\alpha}{2}$ iz brojnika i $2\cos\frac{\alpha}{2}$ iz nazivnika gornjeg razlomka. Racunam:

$$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha+\cos\alpha} = \frac{2\sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)}{2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)}$$

Zamijenim poredak clanova sume u zagradi nazivnika:

$$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha+\cos\alpha} = \frac{2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)}{2\cos\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)}$$

Pokratim sto se pokratiti dade:

$$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha+\cos\alpha} = \frac{{}^1\cancel{2}\sin\frac{\alpha}{2}\cancel{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)}^{\,1}}{{}_1\cancel{2}\cos\frac{\alpha}{2}\cancel{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)}_{\,1}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

No imajuci na umu da vrijedi osnovni trignometrijski identitet za tangens $\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$ gornji izraz prelazi u:

$$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha+\cos\alpha} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \operatorname{tg}\frac{\alpha}{2}$$

Dakle izracunali smo da vrijedi:

$$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha+\cos\alpha} = \operatorname{tg}\frac{\alpha}{2}$$

Pogledamo li desnu stranu izraza iz zadatka vidimo da je ona identicna onome sto smo dobili pa je time zadatak rijesen.

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Zadatak 14: (str. 73) 9) Dokazi identitet:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}=\sin\alpha$$

Rjesenje: Zadatak nam je pokazati da je lijeva strana izraza jednaka desnoj pa u tu svrhu pokusajmo raspisati lijevu stranu izraza:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}$$

U tu svrhu prisjetimo se adicijskih teorema za tangense i kotangense:

$$\operatorname{tg}(\alpha\pm\beta)=\frac{\operatorname{tg}\alpha\pm\operatorname{tg}\beta}{1\mp\operatorname{tg}\alpha\cdot\operatorname{tg}\beta}$$

$$\operatorname{ctg}(\alpha\pm\beta)=\frac{\operatorname{ctg}\alpha\cdot\operatorname{ctg}\beta\mp 1}{\operatorname{ctg}\beta\pm\operatorname{ctg}\alpha}$$

Raspisimo po tim identitetima redom izraze $\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$, $\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$, $\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$, $\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$. Krenimo od $\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$:

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\operatorname{tg}\frac{\pi}{4}+\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}\frac{\pi}{4}\operatorname{tg}\frac{\alpha}{2}}$$

Iscitam iz tablice na prvoj stranici dokumenta da vrijedi $\operatorname{tg}\frac{\pi}{4}=1$. Dakle dalje slijedi:

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\operatorname{tg}\frac{\pi}{4}+\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}\frac{\pi}{4}\operatorname{tg}\frac{\alpha}{2}}=\frac{1+\operatorname{tg}\frac{\alpha}{2}}{1-1\cdot\operatorname{tg}\frac{\alpha}{2}}=\frac{1+\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}\frac{\alpha}{2}}$$

No imajuci na umu da vrijedi osnovni trignometrijski identitet za tangens $\operatorname{tg}\alpha=\frac{\sin\alpha}{\cos\alpha}$ gornji izraz prelazi u:

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{1+\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{1-\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}$$

Svedem izraze u brojniku i nazivniku na zajednicki nazivnik $\cos\frac{\alpha}{2}$:

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{1+\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{1-\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}=\frac{\dfrac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{\dfrac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}$$

Pokratim sto se pokratiti dade:

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\dfrac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cancel{\cos\frac{\alpha}{2}}^{\,1}}}{\dfrac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cancel{\cos\frac{\alpha}{2}}^{\,1}}}=\frac{\dfrac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{1}}{\dfrac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{1}}=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

Dakle dobili smo da vrijedi:

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

Nadalje rapisimo $\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$:

$$\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\operatorname{tg}\frac{\pi}{4}-\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}\frac{\pi}{4}\operatorname{tg}\frac{\alpha}{2}}$$

Iscitam iz tablice na prvoj stranici dokumenta da vrijedi $\operatorname{tg}\frac{\pi}{4}=1$. Dakle dalje slijedi:

$$\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\operatorname{tg}\frac{\pi}{4}-\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}\frac{\pi}{4}\operatorname{tg}\frac{\alpha}{2}}=\frac{1-\operatorname{tg}\frac{\alpha}{2}}{1+1\cdot\operatorname{tg}\frac{\alpha}{2}}=\frac{1-\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}\frac{\alpha}{2}}$$

No imajuci na umu da vrijedi osnovni trignometrijski identitet za tangens $\operatorname{tg}\alpha=\frac{\sin\alpha}{\cos\alpha}$ gornji izraz prelazi u:

$$\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{1-\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{1+\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}$$

Svedem izraze u brojniku i nazivniku na zajednicki nazivnik $\cos\frac{\alpha}{2}$:

$$\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{1-\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{1+\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}=\frac{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}$$

Pokratim sto se pokratiti dade:

$$\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cancel{\cos\frac{\alpha}{2}}^{1}}}{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cancel{\cos\frac{\alpha}{2}}^{1}}}=\frac{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{1}}{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{1}}=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}$$

Dakle dobili smo da vrijedi:

$$\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}$$

Nadalje rapisimo $\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$:

$$\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg}\frac{\alpha}{2}-1}{\operatorname{ctg}\frac{\alpha}{2}+\operatorname{ctg}\frac{\pi}{4}}$$

Iscitam iz tablice na prvoj stranici dokumenta da vrijedi $\operatorname{ctg}\frac{\pi}{4}=1$. Dakle dalje slijedi:

$$\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg}\frac{\alpha}{2}-1}{\operatorname{ctg}\frac{\alpha}{2}+\operatorname{ctg}\frac{\pi}{4}}=\frac{1\cdot\operatorname{ctg}\frac{\alpha}{2}-1}{\operatorname{ctg}\frac{\alpha}{2}+1}=\frac{\operatorname{ctg}\frac{\alpha}{2}-1}{\operatorname{ctg}\frac{\alpha}{2}+1}$$

No imajuci na umu da vrijedi osnovni trignometrijski identitet za kotangens $\operatorname{ctg}\alpha=\frac{\cos\alpha}{\sin\alpha}$ gornji izraz prelazi u:

$$\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-1}{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}+1}$$

Svedem izraze u brojniku i nazivniku na zajednicki nazivnik $\sin\frac{\alpha}{2}$:

$$\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-1}{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}+1}=\frac{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}}{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}}$$

Pokratim sto se pokratiti dade:

$$\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cancel{\sin\frac{\alpha}{2}}^{\,1}}}{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cancel{\sin\frac{\alpha}{2}}^{\,1}}}=\frac{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{1}}{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{1}}=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}$$

Dakle dobili smo da vrijedi:

$$\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}$$

Na kraju rapisimo $\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$:

$$\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg}\frac{\alpha}{2}+1}{\operatorname{ctg}\frac{\alpha}{2}-\operatorname{ctg}\frac{\pi}{4}}$$

Iscitam iz tablice na prvoj stranici dokumenta da vrijedi $\operatorname{ctg}\frac{\pi}{4}=1$. Dakle dalje slijedi:

$$\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg}\frac{\alpha}{2}+1}{\operatorname{ctg}\frac{\alpha}{2}-\operatorname{ctg}\frac{\pi}{4}}=\frac{1\cdot\operatorname{ctg}\frac{\alpha}{2}+1}{\operatorname{ctg}\frac{\alpha}{2}-1}=\frac{\operatorname{ctg}\frac{\alpha}{2}+1}{\operatorname{ctg}\frac{\alpha}{2}-1}$$

No imajuci na umu da vrijedi osnovni trignometrijski identitet za kotangens $\operatorname{ctg}\alpha=\frac{\cos\alpha}{\sin\alpha}$ gornji izraz prelazi u:

$$\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}+1}{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-1}$$

Svedem izraze u brojniku i nazivniku na zajednicki nazivnik $\sin\frac{\alpha}{2}$:

$$\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}+1}{\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-1}=\frac{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}}{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}}$$

Pokratim sto se pokratiti dade:

$$\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cancel{\sin\frac{\alpha}{2}}\,1}}{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cancel{\sin\frac{\alpha}{2}}\,1}}=\frac{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{1}}{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{1}}=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

Dakle dobili smo da vrijedi:

$$\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

Dakle ono sto smo do sada izracunali jest:

$$\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}},\quad \operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}$$

$$\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}},\quad \operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

Vratimo se s time u izraz kojeg smo poceli raspisivati:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}=\frac{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}-\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}}{\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}+\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}}$$

Svedem brojnik i nazivnik gornjeg izraza na zajednicki nazivnik $\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)$. Racunam:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}=\frac{\dfrac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}-\dfrac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}}{\dfrac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}+\dfrac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}}=$$

$$=\frac{\dfrac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2-\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)}}{\dfrac{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2+\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)}}$$

Pokratim sto se pokratiti dade:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}=\frac{\dfrac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2-\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}{\cancel{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)}^{\,1}}}{\dfrac{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2+\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{\cancel{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)}_{\,1}}}=$$

$$=\frac{\dfrac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2-\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}{1}}{\dfrac{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2+\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{1}}=\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2-\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2+\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}$$

Raspisem zagrade u brojniku i nazivniku prema identitetu za kvadrat binoma $(a\pm b)^2=a^2\pm 2ab+b^2$. Slijedi:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}=\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2-\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2+\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}=$$

$$=\frac{\cos^2\frac{\alpha}{2}+2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}-\left(\cos^2\frac{\alpha}{2}-2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}\right)}{\cos^2\frac{\alpha}{2}-2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}+2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}=$$

$$= \frac{\cos^2\frac{\alpha}{2} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} - \cos^2\frac{\alpha}{2} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} - 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}$$

Pokratim sto se pokratiti dade:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) - \operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) + \operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} =$$

$$= \frac{\cancel{\cos^2\frac{\alpha}{2}} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} + \cancel{\sin^2\frac{\alpha}{2}} - \cancel{\cos^2\frac{\alpha}{2}} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} - \cancel{\sin^2\frac{\alpha}{2}}}{\cos^2\frac{\alpha}{2} \cancel{- 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} + \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} \cancel{+ 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} + \sin^2\frac{\alpha}{2}} =$$

$$= \frac{2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{4\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2} + 2\sin^2\frac{\alpha}{2}}$$

Zamijenim poredak clanova umnoska i sume u brojniku i nazivniku, te izlucim dvojku u nazivniku:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) - \operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) + \operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} = \frac{4\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2} + 2\sin^2\frac{\alpha}{2}} = \frac{4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\left(\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\right)}$$

Pokratim sto se pokratiti dade:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) - \operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) + \operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} = \frac{{}^{2}\cancel{4}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{{}_{1}\cancel{2}\left(\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\right)} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}}$$

Uocim da u nazivniku dobivenog razlomka zapravio stoji temeljni trigonometrijski identitet, odnosno da vrijedi $\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1$. Dok s druge strane, ako se prisjetim prethodnog zadatka vrijedi $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$. Imajuci to na umu nastavljam racun:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) - \operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) + \operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} = \frac{\overbrace{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}^{\sin\alpha}}{\underbrace{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}}_{1}} = \frac{\sin\alpha}{1} = \sin\alpha$$

Dakle izracunao sam da vrijedi:

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) - \operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) + \operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} = \sin\alpha$$

Pogledamo li desnu stranu izraza iz zadatka vidimo da je ona identicna onome sto smo dobili pa je time zadatak rijesen.

— ⋆ —